Completeness and Incompleteness of Synchronous Kleene Algebra

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Kleene algebras (KA) can be applied in various contexts, such as relational algebra and automata theory. An important use of KA is as an algebra of regular expressions, where the axioms of KA are used to algebraically characterise equivalences between regular expressions. The axioms of KA correspond well to properties expected of sequential program composition, and hence they provide a logic for reasoning about control flow of sequential programs formulated as regular expressions. Regular languages then provide a canonical semantics for programs expressed as regular expressions, due to a tight connection between regular languages and the axioms of KA: an equation is provable using the KA axioms if and only if the corresponding regular languages coincide [2,3].

Synchronous Kleene algebra (*SKA*) is an extension of KA, and was proposed by Prisacariu to reason about programs that may execute synchronously, i.e., in lock-step [5]. SKA extends KA with a synchronous operator, denoted with $\times$. Synchrony is understood as in Milner's SCCS [4]: each program executes a single action instantaneously at each discrete time step, and all components capable of acting will do so. A semantics for SKA was defined in terms of synchronous languages that resembles the regular language semantics. However, a connection between the axioms of SKA and the synchronous language model such as the one for the axioms of KA and regular languages, turns out to be problematic.

In this talk we first recap some results concerning regular expressions and Kleene algebra. Then we discuss synchronous Kleene algebra as proposed by Prisacariu, and show that its axioms are incomplete w.r.t. the language semantics. Lastly we provide a new axiomatisation and show this axiomatisation is sound and complete w.r.t. Prisacariu's synchronous language model. All these results come from [7](accepted at MPC 2019).

Kleene Algebra

Fix a finite alphabet Σ and let $\mathcal{P}$ denote the powerset. The set of *regular expressions*, denoted $\mathcal{T}_{\mathsf{KA}}$, describes simple programs and is given by the grammar:

$$\mathcal{T}_{\mathsf{KA}} \ni e, f ::= 0 \mid 1 \mid a \in \Sigma \mid e + f \mid e \cdot f \mid e^*$$

The canonical semantics of regular expressions is given in terms of languages. A *word* formed over Σ is a finite sequence of symbols from Σ. The *empty word* is denoted by ε. We write Σ^* for the set of all words over Σ. *Concatenation* of words $u, v \in \Sigma^*$ is denoted by $uv \in \Sigma^*$. A *language* is a set of words. For $K, L \subseteq \Sigma^*$, we define

$$K \cdot L = \{uv : u \in K, v \in L\} \qquad K + L = K \cup L \qquad K^* = \bigcup_{n\in\mathbb{N}} K^n,$$

where $K^0 = \{\varepsilon\}$ and $K^{n+1} = K \cdot K^n$.

To map a regular expression to a language we define $[\![-]\!]_{\mathsf{KA}} : \mathcal{T}_{\mathsf{KA}} \to \mathcal{P}(\Sigma^*)$ inductively:

$$[\![0]\!]_{\mathsf{KA}} = \emptyset \qquad [\![a]\!]_{\mathsf{KA}} = \{a\} \qquad [\![e \cdot f]\!]_{\mathsf{KA}} = [\![e]\!]_{\mathsf{KA}} \cdot [\![f]\!]_{\mathsf{KA}}$$
$$[\![1]\!]_{\mathsf{KA}} = \{\varepsilon\} \qquad [\![e + f]\!]_{\mathsf{KA}} = [\![e]\!]_{\mathsf{KA}} + [\![f]\!]_{\mathsf{KA}} \qquad [\![e^*]\!]_{\mathsf{KA}} = [\![e]\!]_{\mathsf{KA}}^*$$

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A language L is called *regular* if and only if $L = [\![e]\!]_{\mathsf{KA}}$ for some $e \in \mathcal{T}_{\mathsf{KA}}$.

We need rules to determine when two syntactic expressions $e, f \in \mathcal{T}_{\mathsf{KA}}$ are equivalent. For this we take the Kleene algebra axioms, which include for instance associativity of $\cdot$ and commutativity of $+$. We write $\equiv_{\mathsf{KA}}$ for the smallest congruence on $\mathcal{T}_{\mathsf{KA}}$ induced by the Kleene algebra axioms. Intuitively, $e \equiv_{\mathsf{KA}} f$ means that the regular expressions e and f can be proved equivalent according to the axioms of Kleene algebra. We will not list all KA axioms but two axioms are important for this abstract: the *least fixpoint axioms* [2]. They stipulate that for $e, f, g \in \mathcal{T}_{\mathsf{KA}}$ we have

$$e + f \cdot g \leqq_{\mathsf{KA}} g \implies f^* \cdot e \leqq_{\mathsf{KA}} g \qquad\qquad e + f \cdot g \leqq_{\mathsf{KA}} f \implies e \cdot g^* \leqq_{\mathsf{KA}} f$$

where $e \leqq_{\mathsf{KA}} f$ is a shorthand for $e + f \equiv_{\mathsf{KA}} f$. Intuitively, these axioms state that for instance $f^* \cdot e$ is the least solution for g in the equation $e + f \cdot g \leqq_{\mathsf{KA}} g$.

The regular language semantics is called canonical for a reason: a pivotal result in the study of Kleene algebras tells us that $[\![-]\!]_{\mathsf{KA}}$ characterises $\equiv_{\mathsf{KA}}$, in the following sense:

Theorem 1 (Soundness and Completeness of KA [2,3]). *For all $e, f \in \mathcal{T}_{\mathsf{KA}}$, we have that $e \equiv_{\mathsf{KA}} f$ if and only if $[\![e]\!]_{\mathsf{KA}} = [\![f]\!]_{\mathsf{KA}}$.*

Remark 2. The above is equivalent to a more standard formalisation of soundness and completeness. If we have a Kleene algebra with carrier set A, then for any $\sigma : \Sigma \to A$, and any $e, f \in \mathcal{T}_{\mathsf{KA}}$ we have $e \equiv_{\mathsf{KA}} f$ if and only if $\sigma(e) = \sigma(f)$.

Synchronous Kleene Algebra

We now recall synchronous Kleene algebra analogously: we present the set of synchronous regular terms, the language semantics and the existing axioms of SKA [5].

The set of *synchronous regular expressions*, denoted $\mathcal{T}_{\mathsf{SKA}}$, is two-sorted. We first generate a subset of $\mathcal{T}_{\mathsf{SKA}}$, which we will refer to as the *semilattice terms*, denoted $\mathcal{T}_{\mathsf{SL}}$:

$$\mathcal{T}_{\mathsf{SL}} \ni e, f ::= a \in \Sigma \mid e \times f$$

$\mathcal{T}_{\mathsf{SKA}}$ is then given by the grammar:

$$\mathcal{T}_{\mathsf{SKA}} \ni e, f ::= 0 \mid 1 \mid a \in \mathcal{T}_{\mathsf{SL}} \mid e + f \mid e \cdot f \mid e \times f \mid e^*$$

The semantics of synchronous Kleene algebra is similar to the semantics of regular expressions, but instead of a word consisting of a sequence of actions, each words consists of a sequence of *sets* of actions. Thus, the semantics of synchronous regular expressions is given in terms of synchronous languages, and synchronous languages consist of words formed over the alphabet $\mathcal{P}(\Sigma) \setminus \{\emptyset\} = \mathcal{P}_n(\Sigma)$. The standard language operations (sum, concatenation, Kleene closure) are also defined on synchronous languages. The synchronous product of synchronous languages K, L is given by:

$$K \times L = \{u \times v : u \in K, v \in L\}$$

where we define $\times$ inductively for $u, v \in (\mathcal{P}_n(\Sigma))^*$ and $x, y \in \mathcal{P}_n(\Sigma)$, as follows:

$$u \times \varepsilon = u = \varepsilon \times u \qquad \text{and} \qquad (x \cdot u) \times (y \cdot v) = (x \cup y) \cdot (u \times v)$$

To define the language semantics for all elements in $\mathcal{T}_{\mathsf{SKA}}$, we first give an interpretation of elements in $\mathcal{T}_{\mathsf{SL}}$ in terms of non-empty finite subsets of Σ. For $a \in \Sigma$ and $e, f \in \mathcal{T}_{\mathsf{SL}}$, define $[\![-]\!]_{\mathsf{SL}} : \mathcal{T}_{\mathsf{SL}} \to \mathcal{P}_n(\Sigma)$ by

$$[\![a]\!]_{\mathsf{SL}} = \{a\} \qquad\qquad [\![e \times f]\!]_{\mathsf{SL}} = [\![e]\!]_{\mathsf{SL}} \cup [\![f]\!]_{\mathsf{SL}}$$

We define the mapping $[\![-]\!]_{\mathsf{SKA}} : \mathcal{T}_{\mathsf{SKA}} \to \mathcal{P}((\mathcal{P}_n(\Sigma))^*)$ as follows:

$$[\![0]\!]_{\mathsf{SKA}} = \emptyset \quad [\![1]\!]_{\mathsf{SKA}} = \{\varepsilon\} \qquad [\![a]\!]_{\mathsf{SKA}} = \{[\![a]\!]_{\mathsf{SL}}\} \quad \forall a \in \mathcal{T}_{\mathsf{SL}} \qquad [\![e^*]\!]_{\mathsf{SKA}} = [\![e]\!]^*_{\mathsf{SKA}}$$
$$[\![e \cdot f]\!]_{\mathsf{SKA}} = [\![e]\!]_{\mathsf{SKA}} \cdot [\![f]\!]_{\mathsf{SKA}} \qquad [\![e + f]\!]_{\mathsf{SKA}} = [\![e]\!]_{\mathsf{SKA}} + [\![f]\!]_{\mathsf{SKA}} \qquad [\![e \times f]\!]_{\mathsf{SKA}} = [\![e]\!]_{\mathsf{SKA}} \times [\![f]\!]_{\mathsf{SKA}}$$

The synchronous language model can easily be proved to be an SKA (the semilattice S of the language model is defined as all synchronous languages with one word of length one).

The axiomatisation for synchronous regular terms consists of the axioms of Kleene algebra and additional axioms for the synchronous operator. We write $\equiv_{\mathsf{SKA}}$ for the smallest congruence on $\mathcal{T}_{\mathsf{SKA}}$ induced by the SKA axioms, i.e., such that $\equiv_{\mathsf{KA}}$ is included in $\equiv_{\mathsf{SKA}}$ and for all $e, f, g \in \mathcal{T}_{\mathsf{SKA}}$ and $\alpha, \beta \in \mathcal{T}_{\mathsf{SL}}$ the following hold:

$$e \times (f + g) \equiv_{\mathsf{SKA}} e \times f + e \times g \qquad e \times (f \times g) \equiv_{\mathsf{SKA}} (e \times f) \times g \qquad \alpha \times \alpha \equiv_{\mathsf{SKA}} \alpha$$
$$(\alpha \cdot e) \times (\beta \cdot f) \equiv_{\mathsf{SKA}} (\alpha \times \beta) \cdot (e \times f) \qquad e \times 0 \equiv_{\mathsf{SKA}} 0 \quad e \times 1 \equiv_{\mathsf{SKA}} e \qquad e \times f \equiv_{\mathsf{SKA}} f \times e$$

The synchronous product has some expected axioms such as associativity, commutativity and distributivity over $+$. The axiom $(\alpha \cdot e) \times (\beta \cdot f) = (\alpha \times \beta) \cdot (e \times f)$ captures the lock-step behaviour of SKA, and will be referred to as the *synchrony axiom*. Note that in any SKA the synchrony axiom and idempotence of $\times$ are only valid if α and β come from the semilattice structure ($\mathcal{T}_{\mathsf{SL}}$ in this case). Crucially, there is no axiom that describes the interaction between the Kleene star (used to describe loops) and the synchronous product, which we will see turned out to be problematic.

The problem: incompleteness of SKA

It is desirable to link the axioms of SKA to the synchronous language model in a precise way, similar as for Kleene algebras: an equation should be provable from the synchronous Kleene algebra axioms if and only if the corresponding synchronous languages coincide (soundness and completeness). Soudness follows easily [5], and can be generalised analogously to Remark 2: for any SKA with carrier set A and semilattice $S \subseteq A$, for any $\sigma : \Sigma \to S$, we have that for all $e, f \in \mathcal{T}_{\mathsf{SKA}}$ such that $e \equiv_{\mathsf{SKA}} f$ it holds that $\sigma(e) = \sigma(f)$. However, the axioms of SKA are not complete: we found an example of two synchronous languages that coincide in the synchronous language model, yet their corresponding SKA expressions are not provably equivalent. This invalidates the completeness result in [5].

To prove that SKA is incomplete with respect to the model of synchronous languages, we need two expressions $e, f \in \mathcal{T}_{\mathsf{SKA}}$ such that $[\![e]\!]_{\mathsf{SKA}} = [\![f]\!]_{\mathsf{SKA}}$ but $e \not\equiv_{\mathsf{SKA}} f$. To this end, we have built a countermodel exploiting the fact that there are no axioms dictating the interactions between the Kleene star and the synchronous product. For $a \in \Sigma$, we have $[\![a^* \times a^*]\!]_{\mathsf{SKA}} = [\![a^*]\!]_{\mathsf{SKA}}$, but via the countermodel we demonstrate that $a^* \times a^* \not\equiv_{\mathsf{SKA}} a^*$.

Our countermodel $\mathcal{M}$ was constructed to be an SKA such that for all elements α in the semilattice S of $\mathcal{M}$ we have $\alpha^* \times \alpha^* \neq \alpha^*$. Let us consider the SKA term algebra over alphabet Σ. Then suppose that $a^* \times a^* \equiv_{\mathsf{SKA}} a^*$ for $a \in \Sigma$. Using soundness, this entails that under any assignment $\sigma : \Sigma \to S$ assigning a to an element in the semilattice of $\mathcal{M}$, we must have $\sigma(a)^* \times \sigma(a)^* = \sigma(a)^*$. As we have that $\alpha^* \times \alpha^* \neq \alpha^*$ for all $\alpha \in S$ in our countermodel, we have a contradiction, and thus obtain $a^* \times a^* \not\equiv_{\mathsf{SKA}} a^*$.

The solution: a new axiomatisation

We now propose a set of axioms that are in fact sound and complete with respect to the synchronous language model. The key difference with [5] is that we delete the *least* fixpoint axioms in the style of Kozen [2] and add two new axioms in the style of Salomaa [6], the *loop tightening* and *unique fixpoint axiom*:

$$(e+1)^* = e^* \qquad\qquad \epsilon \notin [\![e]\!]_{\mathsf{SKA}} \wedge e \cdot f + g = f \implies e^* \cdot g = f$$

The condition $\epsilon \notin [\![e]\!]_{\mathsf{SKA}}$ can be characterised algebraically [7], but this is omitted for the sake of brevity. The axioms of KA *minus* the least fixpoint axioms and *plus* these two new axioms, which together we call the axioms of F_1-algebra, are shown to be sound and complete with respect to the regular language model by Salomaa [6]. The axioms of Salomaa are strictly stronger than Kozen's [1], and we will see that the unique fixpoint axiom allows us to derive a connection between the synchronous product and the Kleene star, even though this connection is not represented in an axiom directly (see Example 3).

Our proposed set of axioms, which we refer to as the axioms of SF_1-algebra, then contains the axioms of synchronous Kleene algebra *without* the least fixpoint axioms, and the loop tightening and unique fixpoint axiom. We write $\equiv_{\mathsf{SF}_1}$ for the smallest congruence on $\mathcal{T}_{\mathsf{SKA}}$ that satisfies these axioms.

Example 3. To demonstrate the use of the new axioms, we give an algebraic proof of $\alpha^* \times \alpha^* \equiv_{\mathsf{SF}_1} \alpha^*$ for $\alpha \in \mathcal{T}_{\mathsf{SL}}$. It is not hard to derive that $\alpha^* \times \alpha^* \equiv_{\mathsf{SF}_1} \alpha^* + \alpha \cdot (\alpha^* \times \alpha^*)$. Since $\epsilon \notin [\![\alpha]\!]_{\mathsf{SKA}}$ for any $\alpha \in \mathcal{T}_{\mathsf{SL}}$, we can apply the unique fixpoint axiom to find $\alpha^* \cdot \alpha^* \equiv_{\mathsf{SF}_1} \alpha^* \times \alpha^*$. In SF_1, we can prove that $\alpha^* \cdot \alpha^* \equiv_{\mathsf{SF}_1} \alpha^*$; hence, we find $\alpha^* \times \alpha^* \equiv_{\mathsf{SF}_1} \alpha^*$.

To prove completeness of our new axiomatisation with respect to the model of synchronous languages, we gave a reduction to the completeness result of Salomaa. In the proof, most effort is spent to show that for every expression $e \in \mathcal{T}_{\mathsf{SKA}}$ we could find an expression $\hat{e}$ such that $e \equiv_{\mathsf{SF}_1} \hat{e}$, and $\hat{e}$ is a term in normal form. Expressions in normal form are such that the synchronous product only appears on elements of $\mathcal{T}_{\mathsf{SL}}$, e.g. for $a, b \in \Sigma$ the term $(a \times (a \times b)) \cdot (a + b)$ is in normal form, but $a \times (a + b)$ is not. In other words, $\hat{e}$ is actually a regular expression over alphabet $\mathcal{T}_{\mathsf{SL}}$.

Skipping over details, the fact that expressions in normal form are regular expressions allowed us to conclude that for $\hat{e}$ and $\hat{f}$ in normal form, if $[\![\hat{e}]\!]_{\mathsf{SKA}} = [\![\hat{f}]\!]_{\mathsf{SKA}}$ then $[\![\hat{e}]\!]_{\mathsf{KA}} = [\![\hat{f}]\!]_{\mathsf{KA}}$. Using Salomaa's completeness result we obtain that $\hat{e}$ and $\hat{f}$ are provably equivalent using the axioms of F_1-algebras. As all of these axioms are contained in the SF_1 axioms, we can conclude that $\hat{e} \equiv_{\mathsf{SF}_1} \hat{f}$, which proves completeness for terms in normal form.

With this result in hand, we can prove completenss for all terms in $\mathcal{T}_{\mathsf{SKA}}$:

Theorem 4 (Completeness). *For all $e, f \in \mathcal{T}_{\mathsf{SKA}}$, $[\![e]\!]_{\mathsf{SKA}} = [\![f]\!]_{\mathsf{SKA}} \Rightarrow e \equiv_{\mathsf{SF}_1} f$.*

Proof. Assume that $[\![e]\!]_{\mathsf{SKA}} = [\![f]\!]_{\mathsf{SKA}}$. We obtain $\hat{e}$ and $\hat{f}$ in normal form such that $e \equiv_{\mathsf{SF}_1} \hat{e}$ and $f \equiv_{\mathsf{SF}_1} \hat{f}$, and thus $[\![\hat{e}]\!]_{\mathsf{SKA}} = [\![\hat{f}]\!]_{\mathsf{SKA}}$ via soundness. From the completeness result for terms in normal form, we obtain $e \equiv_{\mathsf{SF}_1} \hat{e} \equiv_{\mathsf{SF}_1} \hat{f} \equiv_{\mathsf{SF}_1} f$, proving completeness of our axiomatisation.

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